Math 245C Lecture 22 Notes

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1 The Poisson Summation Formula and Integrability of the Fourier Transform

This lecture was given by a guest lecturer.

1.1 The Poisson summation formula

Recall that if $E_k(x) =^{2\pi i k\cdot x}$, then $\{E_k : k \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$. We have also shown the following:

Theorem 1.1. *If $f \in L^1(\mathbb{R}^n)$, then the series $\sum_{k\in\mathbb{Z}^N} \tau_k f$ converges pointwise a.e. and in $L^1(\mathbb{T}^n)$ to a function Pf such that $\|Pf\|_1 \leq \|f\|_1$. Moreover, $\widehat{Pf}(k) = \widehat{f}(k)$.*

We have also shown the following theorem in the $\mathbb{R}^n$ case, but here is the form of the theorem in the $\mathbb{T}^n$ case.

Theorem 1.2 (Hausdorff-Young inequality)**.** *Suppose that $1 \leq p \leq 2$ and q is the the conjugate exponent of p. If $f \in L^p(\mathbb{T}^n)$, then $\widehat{f} \in \ell^q(\mathbb{Z}^n)$, and $\|\widehat{f}\|_{\ell^q(\mathbb{Z}^n)} \leq \|f\|_{L^p(\mathbb{T}^n)}$.*

Theorem 1.3 (Poisson summation formula)**.** *Suppose that $f \in C(\mathbb{R}^n)$ satisfies $|f(x)| \leq C/(1+|x|)^{n+\varepsilon}$ and $|\widehat{f}(\xi)| \leq C/(1+|\xi|)^{n+\varepsilon}$ for some $C, \varepsilon > 0$. Then*

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{2\pi i l\cdot x},$$

where both series converge absolutely and uniformly on $\mathbb{T}^n$. In particular,

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k).$$

Proof. Since $|f(x)| \leq C/(1+|x|)^{n+\varepsilon}$, for all $x \in \mathbb{T}^n$,

$$|f(x+k)| \leq \frac{C}{(1+|x+k|)^{n+\varepsilon}} \leq \frac{C'}{(1+|k|)^{n+\varepsilon}}.$$

Then compare
$$\sum_{k\in\mathbb{Z}} \frac{C}{(1+|k|)^{n+\varepsilon}} \sim \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+\varepsilon}}\,dx.$$
This implies that
$$\sum_{k\in\mathbb{Z}^n} f(x+k) \overset{C(\mathbb{T}^n)}{=} Pf(x)$$
for all $x\in\mathbb{T}^n$.

By the previous theorem, we have $Pf\in L^1(\mathbb{T}^n)$ and $\widehat{Pf}(k)=\widehat{f}(k)$. Then $Pf\in L^2(\mathbb{T}^n)$, and since $\{E_k : k\in\mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$, we have
$$Pf \overset{L^2(\mathbb{T}^n)}{=} \sum_{k\in\mathbb{Z}^n} \widehat{Pf} e^{2\pi i k\cdot x} = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{2\pi i k\cdot x}.$$
By the decay of $\widehat{f}$, $Pf(x) \overset{C(\mathbb{T}^n)}{=} \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)e^{2\pi i k\cdot x}$. □

1.2 Integrability of the Fourier transform

The Fourier inversion theorem shows how to use $\widehat{f}$ to represent f is $f,\widehat{f}\in L^1(\mathbb{R}^n)$. In $\mathbb{T}^n$, if $f\in L^1(\mathbb{T}^n)$ and $\widehat{f}\in \ell^1(\mathbb{Z}^n)$, then the Fourier series
$$\sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{2\pi i k\cdot x}$$
converges absolutely and uniformly to a function g. Since $\ell^1\subseteq \ell^2$, it follows that $f\in L^2$ and the serires converges to f in L^2. Hence, $f=g$ a.e. We have 2 questions:

1. Under what conditions is $\widehat{f}$ integrable?
2. How can f be recovered from $\widehat{f}$ if $\widehat{f}$ is not integrable?

Theorem 1.4. *Suppose that f is periodic and absolutely continuous on $\mathbb{R}$, and $f'\in L^p(\mathbb{T})$ for some $p>1$. Then $\widehat{f}\in\ell^1(\mathbb{Z})$.*

Proof. By integration by parts, $\widehat{f'}(k)=2\pi i k\widehat{f}(k)$. Hence, by Hölder's inequality,
$$\begin{aligned}\sum_{k\neq 0}|\widehat{f}(k)| &\le \underbrace{\left(\sum_k (2\pi|k|)^{-p}\right)^{1/p}}_{=:C_p}\left[\sum_{k\neq 0}(2\pi|k\widehat{f}(k)|^q)^{1/q}\right] \\ &= C_p\left(\sum_{k\neq 0}|\widehat{f'}(k)|^q\right)^{1/p}\end{aligned}$$

$$\leq C_p \|\widehat{f'}\|_{\ell^q(\mathbb{Z})}.$$

Since $L^p(\mathbb{T}) \subseteq L^2(\mathbb{T})$ for $p > 2$, we can assume that $1 < p \leq 2$. By the Hausdorff-Young inequality,

$$\sum_{k \neq 0} |\widehat{f}(k)| \leq C_p \|f'\|_{L^p(\mathbb{T})}.$$

Adding $|\widehat{f}(0)|$ to both sides, we see that

$$\|\widehat{f}\|_{\ell^1(\mathbb{Z})} < \infty. \qquad \square$$

Lemma 1.1. *If $f, g \in L^2(\mathbb{R}^n)$, then $(\widehat{f}\widehat{g})^\vee = f * g$.*

Proof. By assumption, we know that $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}^n)$. Then $\widehat{f}\widehat{g} \in L^1(\mathbb{R}^n)$. So $(\widehat{f}\widehat{g})^\vee$ makes sense. So for $x \in \mathbb{R}^n$, define $h(y) = \overline{g(x-y)}$. Then $\widehat{h}(\xi) = \overline{\widehat{g}(\xi)} e^{-2\pi i \xi \cdot x}$. Then

$$f * g(x) = \int_{\mathbb{R}^n} f\overline{h} = \int \widehat{f}\overline{\widehat{h}} = \int \widehat{f}(\xi)\widehat{g}(\xi e^{2\pi i \xi \cdot x}\, d\xi = (\widehat{f}\widehat{g})^\vee. \qquad \square$$